Indian Statistical Institute, Bangalore
M. Math.I Year, First Semester
Semestral Examination   Measure Theoretic Probability   Time: 3 hours

Instructor: C.R.E Raja   November 26th, 2009   Maximum Marks 50

Section I - Total Marks 10

Answer all and each question is worth 2 marks

1. Let X be a uncountable set and $\mathcal{A} = \{E \subset X \mid E \text{ or } X \setminus E \text{ is countable }\}$. Show that $\mathcal{A}$ is a σ-algebra.

2. Let f be a real-valued Lebesgue measurable function on a closed bounded interval $[a, b]$. Given $\epsilon > 0$, show that there is a $M > 0$ such that $|f(x)| \leq M$ except on a set of Lebesgue measure less than ϵ.

3. For a r.v. X, show that X and e^X are independent if and only if X degenerates.

4. For any probability measures μ and λ on $\mathbb{R}$, prove that $\mu * \lambda$ has an atom implies μ and λ has atoms.

5. Let (μ_n) and (λ_n) be sequences of probability measures on $\mathbb{R}$ converging weakly to probability measures μ and λ respectively. Then prove that $(\mu_n * \lambda_n)$ converges weakly to $\mu * \lambda$.

Section II - Total Marks 20
Answer any four and each question is worth 5 marks

1. Prove that any open interval is Lebesgue measurable.

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f : X \to \mathbb{C}$ be an integrable function on X. Suppose $S \subset \mathbb{C}$ is a closed set such that $A_E(f) = \frac{1}{\mu(E)} \int_E f \in S$ for any $E \in \mathcal{A}$ with $\mu(E) > 0$. Then prove that $f(x) \in S$ for a.e. x in X.

3. Let $(X, \mathcal{A})$ be a measurable space and μ be a σ-finite measure. Show that there exists a probability measure λ on X such that λ and μ are absolutely continuous with respect to each other and find $\frac{d\lambda}{d\mu}$.

4. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and (E_n) be independent events in $\mathcal{A}$. Prove that $\sum \mathcal{P}(E_n) = \infty$ implies $\mathcal{P}(E_n \text{ i.o.}) = 1$.

5. If (X_n) is a sequence of independent r.v.'s on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, find the possible values of $\mathcal{P}(X_n \text{ converges or diverges })$ and justify.

6. Let ρ and λ be probability measures on $\mathbb{R}$. Suppose (λ^n) converges weakly to ρ. Then prove that $\rho * \rho = \rho$ and deduce that $\rho = \delta_0 = \lambda$.

Section III - Total marks 20
Answer any two and each question is worth 10 marks

1. (a) Let g be a Lebesgue measurable function on a Lebesgue measurable set E and f be a real-valued function on E such that $f = g$ a.e. Then show that f is also Lebesgue measurable.

   (b) Let f be a bounded function on a Lebesgue measurable set E. Suppose there exist two sequences (f_n) and (g_n) of Lebesgue integrable functions on E such that $-\infty < f_n \leq f \leq g_n < \infty$ on E and $\int_E [g_n(x) - f_n(x)]dx \leq \frac{1}{2^n}$ for all $n \geq 1$. Then show that f is Lebesgue measurable.

2. Let (X_n), (Y_n) be sequences of r.v.'s on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$.

   (a) Prove that $X_n \to 0$ a.e. if and only if $\mathcal{P}(|X_n| > \epsilon \text{ i.o.}) = 0$ for any $\epsilon > 0$

   (b) If $X_n \to 0$ and $Y_n \to 0$ in probability, then show that the sequences (X_n+Y_n), $(X_n - Y_n)$ and (X_nY_n) converge to 0 in probability.

3. (a) Let μ be a probability measure on $\mathbb{R}$ and A be the set of atoms of μ. Prove that A is countable and to each $\epsilon > 0$, there are $x, y \notin A$ such that $\mu([x, y]) > 1 - \epsilon$.

   (b) Suppose a sequence (μ_n) of probability measures on $\mathbb{R}$ converge weakly to a probability measure μ on $\mathbb{R}$. Then show that $\hat{\mu}_n \to \hat{\mu}$ uniformly on compact sets in $\mathbb{R}$.